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A CHARACTERIZATION OF VECTOR MEASURE GAMES IN *pNA*†

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ABSTRACT

We give a complete characterization of games in pNA of the form $f \circ \mu$ (where μ is a vector of finite number of non-atomic probability measures, and f is a real valued function on the range of μ with $f(0)=0$). Specifically, we show that $f \circ \mu$ is in pNA iff "f is continuous at μ" (the definition of the latter is given in the paper).

1. Introduction

We begin with a few definitions, taken from Aumann and Shapley's book [2].

Let $(I, \mathscr{C})$ be a measurable space which is isomorphic to $([0,1], \mathscr{B})$ (the unit interval with its Borel subsets). A *set function* (or *game* v) is a real valued function on $\mathscr{C}$ such that $v(\emptyset)=0$. For each game v define $\|v\|_{BV}$ by $\|v\|_{BV} = \sup\|v\|_\Omega$, where the sup ranges over all chains Ω of the form $\Omega : \emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = I$, $S_i \in \mathscr{C}$ and $\|v\|_\Omega = \sum_{i=0}^{n-1} |v(S_{i+1}) - v(S_i)|$. Denote by BV the set of all games v with $\|v\|_{BV} < \infty$. $(BV, \|\ \|_{BV})$ is a norm space (even Banach one). Denote by NA the set of all non-atomic measures on $(I, \mathscr{C})$, by NA^1 the set of all probability measures in NA, and by pNA the closed linear subspace of BV spanned by all powers of NA^1 measures.

The space pNA plays a central role in the theory of non-atomic games. In [2], Aumann and Shapley proved the existence of a unique value on pNA and presented a formula which enables us to compute the value for games in pNA of the form $f \circ \mu$, where μ is a finite vector of NA measures, and f is a real function defined on the range of μ, with $f(0)=0$. Therefore it is natural to ask which games of the form $f \circ \mu$ are in pNA?

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Another motivation for characterizing the games $f \circ \mu$ in pNA arises from market games. Those games were first treated deeply by Aumann and Shapley in Chapter 6 (*An Application of Economic Equilibrium*) of [2], and then by Aumann and Kurz [1], by S. Hart [5] and by others. The prominence of market games in pNA is illustrated by the fact that every such game has a unique member in the core which coincides with the value of v and with the unique competitive payoff distribution (see [2, proposition 32.3], p. 186).

In fact, the games of the form $f \circ \mu$, where μ is a vector of finitely many measures in NA, and f is continuous, concave and homogeneous of degree 1 on the range of μ, are the market games of finite type where the utility functions are not necessarily differentiable. Therefore a characterization of games $f \circ \mu$ in pNA leads to a characterization of all market games in pNA of finite type.

A characterization for the case where μ is a scalar measure in NA^1 is given in [2]:

THEOREM C [2, p. 25]. *Let $\mu \in NA^1$ and let f be a real function on $[0,1]$ with $f(0)=0$. Then $f \circ \mu$ is in pNA if and only if f is absolutely continuous on $[0,1]$.*

A characterization for the case where μ is a scalar signed measure in NA has been given by E. Kohlberg [6]. He proved the following

THEOREM. *Let μ be in NA with range $[-a,b]$ where $-a<0<b$; let f be a real function on $[-a,b]$ with $f(0)=0$. Let g be the function defined for each $x \neq y$ by*

$$g(x,y)=(x+a)(y-b)\,\frac{f(y)-f(x)}{y-x}.$$

Then $f \circ \mu$ is in pNA if and only if f is continuously differentiable on the open interval $(-a,b)$, $g(x,y)\to 0$ as $x\to -a$ and $y \to b$.

In the case where μ is a finite vector of measures in NA^1 no characterization has heretofore been known. In [2] the authors introduce two sufficient conditions (neither of them necessary) and one necessary condition (which is not sufficient) for $f \circ \mu$ to be in pNA. The sufficient conditions are

THEOREM B [2, p. 23]. *$f \circ \mu$ is in pNA if f is continuously differentiable on the range of μ, and $f(0)=0$.*

PROPOSITION 10.17 [2, p. 92]. *$f \circ \mu$ is in pNA if f is a continuous and non-decreasing function in R^n_+ with $f(0)=0$, and for each $1 \leqq i \leqq n$ the derivative $\partial f/\partial x_i$ exists and is continuous whenever $x_i>0$.*

The necessary condition, which follows from proposition 24.1 of [2] (p. 157), is as follows:

THEOREM. *If $f \circ \mu$ is in pNA, then almost everywhere along the diagonal $[\mu(0), \mu(I)]$, f has, for each $S \in \mathscr{C}$, a derivative in the direction of $\mu(S)$.*

In this paper we introduce a complete characterization of games $f \circ \mu$ (μ is a vector of finitely many measures in NA^1) in pNA. For that purpose we need a few more definitions.

Define a norm $\| \ \|_m$ on the linear space NA^m (the set of all vectors of NA measures having m components) by

$$\mu = (\mu_1, \cdots, \mu_m) \Rightarrow \|\mu\|_m = \sum_{i=1}^{m} \|\mu_i\|_{BV}.$$

Let μ be in NA^m. Denote by $R(\mu)$ the range of μ, i.e.,

$$R(\mu) = \{\mu(S) \mid S \in \mathscr{C}\}.$$

Define $B(\mu)$ and $B(\mu, \varepsilon)$ for each $\varepsilon > 0$ by

$$B(\mu) = \{\nu \in NA^m \mid R(\nu) = R(\mu)\},$$

and

$$B(\mu, \varepsilon) = \{\nu \in B(\mu) \mid \|\mu - \nu\|_m < \varepsilon\}.$$

Fix μ in $(NA^1)^m$, and fix a real function f on $R(\mu)$ with $f(0) = 0$. Define an operator $T_f : (B(\mu), \| \ \|_m) \to (BV, \| \ \|_{BV})$ by

$$T_f(\nu) = f \circ \nu.$$

DEFINITION 1. We will say that "f is continuous at μ" if T_f is continuous at μ. In other words "f is continuous at μ" if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\nu \in B(\mu, \delta) \Rightarrow \|f \circ \mu - f \circ \nu\|_{BV} < \varepsilon.$$

We are ready to state our main result.

THEOREM. *A necessary and sufficient condition for $f \circ \mu$ to be in pNA is that f is continuous at μ.*

In the course of the proof of the theorem two other properties of vector measure games in pNA are established:

PROPERTY I. If $f \circ \mu$ is in pNA then there exists a sequence of polynomials $(p_n)_{n=1}^{\infty}$, all of them on R^m, such that $\|p_n \circ \mu - f \circ \mu\|_{BV} \xrightarrow[n\to\infty]{} 0$.

PROPERTY II. If $f \circ \mu$ is in pNA then $f \circ \nu \in pNA$ for each $\nu \in B(\mu)$. (Therefore T_f is continuous at each point of $B(\mu)$.)

2. The proof of the theorem

We will first prove the theorem for the case where the range of μ, $R(\mu)$, has full dimension, i.e., where $R(\mu)$ contains a ball in R^m. (Recall that $\mu \in (NA^1)^m$, and $f: R(\mu) \to R^\dagger$ with $f(0)=0$, are fixed.)

a. *The condition is sufficient*

Let us start with the idea of the proof. Assuming that f is continuous at μ, we will prove that $f \circ \mu$ is in pNA by showing that one can approximate $f \circ \mu$ by games $g \circ \mu$ where g is continuously differentiable on the range $R(\mu)$ of μ and $g(0)=0$. For that purpose we will smooth f by averaging it, at each point x (in $R(\mu)$), over a small cube close to x. More precisely, for each $0<\delta<1$ and $x \in R(\mu)$, define $f^\delta(x)$ by

$$(*) \qquad f^\delta(x) = \frac{1}{a^m}\int_{y\in C} f((1-\delta)x+\delta y)d\lambda(y) - \frac{1}{a^m}\int_{y \in C} f(\delta y)d\lambda(y)$$

where C is a cube of volume a^m, contained in the interior of $R(\mu)$, and λ is the Lebesgue measure on R^m. (We subtract the term $(1/a^m)\int_{y\in C} f(\delta y)d\lambda(y)$ in the definition of f^δ to obtain $f^\delta(0)=0$.) Of course we first have to justify the definition of f^δ by proving that the above integrals exist. Indeed, we prove that f is continuous in the interior of $R(\mu)$ (Lemma 4). Then we prove that f^δ is continuously differentiable on $R(\mu)$ (Lemma 7); and hence from theorem B of [2] we have that $f^\delta \circ \mu \in pNA$. Now, we use Lemmas 8 and 9 below, and the fact that f is continuous at μ, to prove that $\|f^\delta \circ \mu - f \circ \mu\|_{BV} \to 0$ as $\delta \to 0$. Together with the closedness of pNA (in the variation norm) we then obtain that $f \circ \mu$ is in pNA, as claimed.

LEMMA 2. *For a given η in $(NA^1)^m$ and x in the relative interior of $R(\eta)$, the following holds: For each $\varepsilon>0$ there exists $\delta>0$ such that for each $y \in R(\mu)$ with*[††] *$\|y-x\|<\varepsilon$, there are T and S in $\mathscr{C}$ such that*

$$\eta(S)=x, \qquad \eta(T)=y, \qquad \|\eta(S \,\triangle\, T)\|<\varepsilon.$$

[†] In the variation norm.
[††] On an Euclidean space, $\|\ \ \|$ denotes the Euclidean norm.

PROOF. First we will prove the lemma for the case where $R(\eta)$ has full dimension. Let $\varepsilon > 0$ be given and let M be a real number satisfying $\|z\| \leqq M$ for each $z \in R(\eta)$. There exists $\varepsilon_1 > 0$ such that the ball $B(x, \varepsilon_1)$ (with center x and radius ε_1) is contained in $R(\eta)$. W.l.o.g. (without loss of generality) we can assume that $\|x\| \leqq 1$.

(a) For each ε_2, $0 < \varepsilon_2 < \varepsilon_1$ and for each $z \in E^m$ with $\|z\| < \varepsilon_2$ there exists x_1 and y_1 in $(\varepsilon_2/2\varepsilon_1)R(\eta)$ for which $z = y_1 - x_1$. Figure 1 illustrates the situation. A and B are the intersection points of $\partial B(x, \varepsilon_1)$ with the line connecting x and $x + z$. The points x_1 and y_1 are on the intervals $[0, A]$ and $[0, B]$, respectively, and the line containing the interval $[x_1, y_1]$ is parallel to the one containing $[A, B]$. If $t = \|z\|/2\varepsilon_1$ then $x_1 = t \cdot A$ and $y_1 = tB$. Therefore x_1 and y_1 are in $(\|z\|/2\varepsilon_1)R(\eta)$; and since $\|z\| < \varepsilon_2$, x_1 and y_1 are in $(\varepsilon_2/2\varepsilon_1)R(\eta)$.

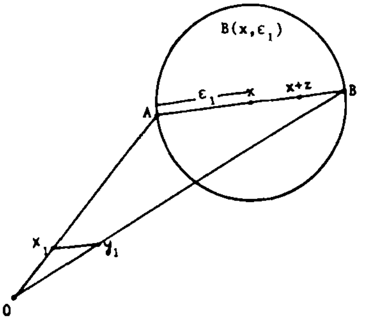

Fig. 1.

(b) Obviously, for each $\varepsilon_4 > 0$ sufficiently small there exists $\varepsilon_3 > 0$ such that $B(x, \varepsilon_3) \subseteq (1 - \varepsilon_4)R(\mu)$. This, of course, remains true if we replace ε_3 by any smaller positive number.

(c) Let us choose $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ such that (b) holds and $\varepsilon_3/\varepsilon_1 + (1 - \varepsilon_4) \leqq 1$.

(d) Choose δ, $0 < \delta < \min(\varepsilon_3, \varepsilon \cdot \varepsilon_1/M)$ small enough such that for each $z \in R^m$

$$z \in \frac{\delta}{2\varepsilon_1} R(\eta) \Rightarrow \|z\| < \varepsilon_3.$$

Let $y \in B(x, \delta)$. Denote $z = y - x$. From (a) and (c) there exist x_1 and y_1 in $(\delta/2\varepsilon_1)R(\eta)$ for which $z = y_1 - x_1$. (d) implies that $\|x_1\| < \varepsilon_3$ and hence $x - x_1 \in B(x, \varepsilon_3)$ and so by (b) $x - x_1 \in (1 - \varepsilon_4)R(\eta)$. Therefore there is $S_3 \in \mathscr{C}$ such that

(e) $f_3 = (1-\varepsilon_4)\cdot \chi_{S_3} \Rightarrow \int_I f_3 d\eta = x - x_1$. (Here χ_{S_3} is the characteristic function of S_3.) Moreover, since x_1 and y_1 are in $(\delta/2\varepsilon_1)\cdot R(\eta)$ there are measurable sets S_1 and S_2 with

(f) $f_2 = (\delta/2\varepsilon_1)\cdot \chi_{S_2} \Rightarrow \int_I f_2 d\eta = x_1$ and

(g) $f_1 = (\delta/2\varepsilon_1)\cdot \chi_{S_1} \Rightarrow \int_I f_1 d\eta = y_1$.

From (c), and the choice of δ, we obtain

(h) $0 \leqq f_3 \leqq f_3 + f_1 \leqq f_3 + f_2 + f_1 \leqq \delta/2\varepsilon_1 + \delta/2\varepsilon_1 + (1-\varepsilon_4) \leqq 1$.

To complete the proof of Lemma 2 we need now the following lemma:

LEMMA 3. *Let η be in NA^m; and let $g_1, g_2, \dots, g_n$ be n measurable functions defined on I, satisfying*

(1) $$0 \leqq g_i(t) \leqq 1 \qquad \text{for each } 1 \leqq i \leqq n \text{ and each } t \in I,$$

(2) $$g_1 \leqq g_2 \leqq \cdots \leqq g_n.$$

Then, there are n measurable sets $T_1, \cdots, T_n$ with $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_n$ such that

$$\eta(T_i) = \int_I g_i d\eta, \qquad i = 1, \cdots, n.$$

Lemma 3 is an immediate consequence of the Dvoretsky–Wald–Wolfowitz theorem [4]. For a proof see lemma 44.1 of [2, p. 259].

From Lemma 3 and (h) we deduce the existence of measurable sets T_1, T_2 and T_3 with $T_3 \subseteq T_1 \subseteq T_2$ and

$$\eta(T_3) = \int_I f_3 d\eta = x - x_1,$$

$$\eta(T_1) = \int_I (f_3 + f_1) d\eta = x - x_1 + y_1,$$

$$\eta(T_2) = \int_I (f_3 + f_1 + f_2) d\eta = x + y_1.$$

Define T and S by

$$T = T_1, \qquad S = T_3 \cup (T_2 \setminus T_1).$$

Then

$$\eta(T) = x - x_1 + y_1 = x + z = y,$$

$$\eta(S) = x \quad \text{and} \quad \eta(T \triangle S) = \eta(T_1 \setminus T_3) + \eta(T_2 \setminus T_1).$$

But

$$\|\eta(T_1\setminus T_3)\| = \|y_1\| < \frac{\delta}{2\varepsilon_1}\cdot M < \frac{\varepsilon}{2}$$

and

$$\|\eta(T_2\setminus T_1)\| = \|x_1\| < \frac{\varepsilon}{2}\,.$$

Hence $\|\eta(T\,\triangle\,S)\| < \varepsilon$ and the proof of Lemma 2 is completed for the case where η has full dimension.

Assume now that l is the dimension of $R(\eta)$ and $l < m$. Since $0 \in R(\eta)$ there is a linear mapping ψ, from R^m onto R^l, which is one to one on the linear space $M(\eta)$ spanned by $R(\eta)$. Let $\tau = \psi\,|_{M(\eta)}$. Define a vector measure ξ of dimension l by $\xi(S) = \tau(\eta(S))$ for each S in $\mathscr{C}$. Let x be in the relative interior of $R(\eta)$, then τx is in $\operatorname{Int} R(\xi)$ (the interior of $R(\xi)$). Now let $\varepsilon > 0$. τ^{-1} is continuous on $R(\xi)$, hence, there is $\beta > 0$ s.t. for each $z \in R(\xi)$

$$\|z\| < \beta \Rightarrow \|\tau^{-1}z\| < \varepsilon. \tag{3}$$

Using the first part of the proof, we have $\delta > 0$ s.t. for each $y \in R(\eta)$ with $\|\tau y - \tau x\| < \delta$ there are T and S in $\mathscr{C}$ for which

$$\tau x = \xi(S), \quad \tau y = \xi(T) \quad \text{and} \quad \|\xi(S\,\triangle\,T)\| < \beta,$$

or

$$x = \eta(S), \quad y = \eta(T) \quad \text{and} \quad \|\xi(S\,\triangle\,T)\| < \beta.$$

Therefore (3) implies $\|\eta(S\,\triangle\,T)\| < \varepsilon$, and the proof of Lemma 2 is complete.

Lemma 4. *f is continuous in* $\operatorname{Int} R(\mu)$.

Proof. Let $\varepsilon > 0$. From the continuity of f at μ, there is $\beta > 0$ s.t. $\nu \in B(\mu, \beta)$ implies

$$\|f\circ\mu - f\circ\nu\|_{BV} < \varepsilon. \tag{4}$$

Let x be in $\operatorname{Int} R(\mu)$. By Lemma 2 there is $\delta > 0$ s.t. for $y \in R(\mu)$ with $0 < \|y - x\| < \delta$ there are measurable sets T and S with

$$x = \mu(S), \quad y = \mu(T) \quad \text{and} \quad 0 < \|\mu(S\,\triangle\,T)\| < \beta/2m.$$

Let us fix now the vector y and the corresponding two sets S and T. W.l.o.g. we can assume that $y \neq \mu(I)$. Denote $\bar{\mu} = \Sigma_{i=1}^m \mu_i$ where $\mu = (\mu_1, \cdots, \mu_m)$ and consider two cases.

I. $\bar{\mu}(T \setminus S)=0$. In this case we choose $S_0 \in \mathscr{C}$ such that

$$S_0 \subseteq (I \setminus T) \setminus S, \quad \bar{\mu}(S_0)=0 \quad \text{and}$$

$$S_0 \text{ and } S \setminus T \text{ have the same cardinality.}$$

We use now the following theorem:

THEOREM 5. *Any uncountable Borel subset of any complete separable metric space, when considered as a measurable space (with the σ-field of the Borel subsets), is isomorphic to* $([0,1], \mathscr{B})$,

(For a proof, see Mackey [7].) Choose an automorphism θ of $(I, \mathscr{C})$ s.t.

$$\theta S_0 = S \setminus T, \quad \theta(S \setminus T) = S_0 \quad \text{and}$$

$$\theta x = x \qquad \text{for each } x \notin S_0 \cup (S \setminus T).$$

Define a vector measure ν by $\nu = \theta * \mu$ (i.e., $\nu(S)=\mu(\theta S)$ for each $S \in \mathscr{C}$). ν is in $B(\mu)$ and

$$\begin{aligned}\|\mu - \nu\|_m &= \sum_{i=1}^{m} \|\mu_i - \nu_i\|_{BV} = \sum_{i=1}^{m} \|\mu_i - \theta * \mu_i\|_{BV} \\ &= \sum_{i=1}^{m} \sup_A \left[\mu_i(A) - \mu_i(\theta A) - \mu_i(A^c) + \mu_i(\theta A^c)\right]\end{aligned}$$

where A^c is the complement of A $(A \in \mathscr{C})$. From the definition of θ we get

$$\begin{aligned}\|\mu - \nu\|_m = \sum_{i=1}^{m} \sup_A \,&[\mu_i(A \cap (S \setminus T)) - \mu_i(\theta A \cap (S \setminus T)) \\ &- \mu_i(A^c \cap (S \setminus T)) + \mu_i(\theta A^c \cap (S \setminus T))].\end{aligned}$$

Hence

$$\begin{aligned}\|\mu - \nu\|_m &\leqq \sum_{i=1}^{m} \sup_A \left[\mu_i(A \cap (S \setminus T)) + \mu_i(\theta A^c \cap (S \setminus T))\right] \\ &\leqq \sum_{i=1}^{m} 2\mu_i(S \setminus T) < 2m \cdot \frac{\beta}{2m} = \beta.\end{aligned}$$

Therefore, by (4) we have $\|f \circ \mu - f \circ \nu\|_{BV} < \varepsilon$, which implies that

$$|f(\mu(T)) - f(\mu(S))| < \varepsilon,$$

or

$$|f(y) - f(x)| < \varepsilon.$$

The proof of the lemma for case I is thus complete. The case where $\bar{\mu}(S \setminus T) = 0$ is analogous.

II. $\bar{\mu}(S \setminus T) > 0$ *and* $\bar{\mu}(T \setminus S) > 0$. Apply again Theorem 5 to get an automorphism θ of $(I, \mathscr{C})$ satisfying

$$\theta(T \setminus S) = S \setminus T, \quad \theta(S \setminus T) = T \setminus S \quad \text{and}$$

$$\theta x = x \qquad \text{for each } x \notin S \triangle T.$$

The proof of Lemma 4 is completed now, in the same fashion as case I.

Definition 6. Let C be the cube in R^m with center $\mu(I)/2$ and side a defined by $C = [-a/2, a/2]^m + \mu(I)/2$, where a is a small positive number for which C is contained in Int $R(\mu)$. For each $0 < \delta < 1$ and $x \in R(\mu)$ define $f^\delta(x)$ by

$$(5) \qquad f^\delta(x) = \frac{1}{a^m}\int_{y \in C} f((1-\delta)x + \delta \cdot y)d\lambda(y) - \frac{1}{a^m}\int_{y \in C} f(\delta \cdot y)d\lambda(y).$$

Notice that f^δ is defined on an open neighborhood D of $R(\mu)$. Also notice that the two integrals in (5) are well defined; since f is continuous on Int $R(\mu)$ and for each $x \in R(\mu)$ and $0 < \delta < 1$ the cube $(1-\delta)x + \delta C$ is contained in Int $R(\mu)$.

Obviously for each $x \in R(\mu)$, $f^\delta(x)$ can be written as

$$(6) f^\delta(x) = \frac{1}{a^m \cdot \delta^m}\int \cdots \int_{(1-\delta)x_i + \delta(1/2 - a/2)}^{(1-\delta)x_i + \delta(1/2 + a/2)} \int f(z)dz_1 \cdots dz_m - \frac{1}{a^m}\int_{y \in C} f(\delta \cdot y)d\lambda(y).$$

Since $(1-\delta)x + \delta \cdot C$ is contained in Int $R(\mu)$, f^δ has continuous partial derivatives at x. But this is the case for every $x \in R(\mu)$, thus f^δ is continuously differentiable on $R(\mu)$.[†] Hence

Lemma 7. *For each $0 < \delta < 1$, f^δ is continuously differentiable on $R(\mu)$.*

Our next purpose is to prove that $\|f^\delta \circ \mu - f \circ \mu\|_{BV} \xrightarrow[\delta \to 0]{} 0$. For this we need the following two lemmas.

Lemma 8. *f is continuous at 0 and $\mu(I)$.*

[†] The concept of continuous differentiability of a real function f, on a convex set X contained in R^n, is defined as follows (see [2, p. 22]): A vector z is said to be X admissible if $z = x - y$ for some x and y in X. Let f be a continuous real function on X. We shall say that f is continuously differentiable on X, if for each X admissible z there is a real function on X which equals the derivative $df(x + hz)/dh$ (this involves the assumption that the derivative exists) at each point x in the relative interior of X, and which is continuous at each point in X.

PROOF. Assume that $x_k \to 0$ as $k \to \infty$, and $x_k \in R(\mu)$ for each k. We choose $S_k \in \mathscr{C}$ s.t. $\mu(S_k) = x_k$. Theorem 5 enables us to select $A_k \in \mathscr{C}$ with $A_k \cap S_k = \varnothing$ and $\mu(A_k) = 0$ together with automorphisms θ_k for which

$$\theta_k S_k = A_k, \quad \theta_k A_k = S_k \quad \text{and}$$

$$\theta_k x = x \qquad \text{for each } x \not\in A_k \cup S_k.$$

For each k, $\theta_k^* \mu \in B(\mu)$ and $\| \mu - \theta_k^* \mu \|_{BV} \xrightarrow[k\to\infty]{} 0$. (The proof is similar to the one given in Lemma 4.) Let $\varepsilon > 0$. f is continuous at μ, hence for large k

$$\| f \circ \mu - f \circ (\theta_k^* \mu) \|_{BV} < \varepsilon,$$

therefore $|f(\mu(S_k)) - f(\mu(A_k))| < \varepsilon$, i.e., $|f(x_k)| < \varepsilon$. So f is continuous at 0. The continuity of f at $\mu(I)$ is proved in a similar way.

LEMMA 9. *Let $\eta \in (NA^1)^m$ and let S_1, S_2 be two measurable sets such that S_1 and $I \setminus S_2$ are both uncountable sets and $S_1 \subseteq S_2$. Then for each $0 < \delta < 1$ and for each $y \in R(\eta)$ there is a vector measure η^y such that*

$$\eta^y \in B(\eta, 2m\delta), \tag{7}$$

$$\eta^y(S) = (1-\delta)\eta(S) + \delta y \qquad \textit{for each } S \in \mathscr{C} \textit{ with } S_1 \subseteq S \subseteq S_2. \tag{8}$$

PROOF. Let A and B be two uncountable subsets of S_1 and $I \setminus S_2$ respectively such that

$$\eta(A) = \eta(B) = 0. \tag{9}$$

Let $y \in R(\eta)$. Choose $S_y \in \mathscr{C}$ such that S_y and $I \setminus S_y$ are both uncountable sets and

$$\eta(S_y) = y. \tag{10}$$

Apply Theorem 5 to have an isomorphism $\theta : A \cup B \to I$ such that

$$\theta A = S_y \quad \text{and} \quad \theta B = I \setminus S_y. \tag{11}$$

For each $0 < \delta < 1$ define the vector measure η^y by

$$\eta^y(S) = \delta \eta(\theta(S \cap (A \cup B))) + (1-\delta)\eta(S), \qquad \text{for each } S \in \mathscr{C}. \tag{12}$$

Since for each $T \subseteq A \cup B$ and each $\bar{T} \subseteq I \setminus (A \cup B)$

$$\eta^y(T) = \delta \eta(\theta T) \quad \text{and} \quad \eta^y(\bar{T}) = (1-\delta)\eta(\bar{T}),$$

we obtain $R(\eta^y) = R(\eta)$. Now by the equation

$$\eta^y(S) - \eta(S) = \delta[\eta(\theta(S \cap (A \cup B))) - \eta(S)]$$

we have

$$\|\eta^y - \eta\| \leqq 2m\delta.$$

Finally, if $S \in \mathscr{C}$ obeys $S_1 \subseteq S \subseteq S_2$ then by (10), (11), and (12)

$$\begin{aligned}
\eta^y(S) &= \delta\eta(\theta(S_1 \cap A)) + (1-\delta)\eta(S) \\
&= \delta\eta(\theta A) + (1-\delta)\eta(S) \\
&= \delta\eta(S_y) + (1-\delta)\eta(S) = \delta y + (1-\delta)\eta(S).
\end{aligned}$$

The proof of Lemma 9 is thus complete.

LEMMA 10. *For each $\varepsilon > 0$, there exists $0 < \delta_0 < 1$ s.t. for every $0 < \delta < \delta_0$*

$$\|f^\delta \circ \mu - f \circ \mu\|_{BV} < \varepsilon.$$

PROOF. Let $\varepsilon > 0$ be given. Because of the continuity of f at μ, there is $r > 0$ s.t. for each $\nu \in B(\mu, r)$

$$\|f \circ \mu - f \circ \nu\|_{BV} < \varepsilon/2. \tag{13}$$

Let Ω be an arbitrary chain $\Omega: \varnothing = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = I$ of measurable sets. W.l.o.g. we can assume that S_1 and $I \setminus S_{k-1}$ are uncountable. It is sufficient to prove that $\|f \circ \mu - f^\delta \circ \mu\|_\Omega < \varepsilon$. For each δ, $0 < \delta ScQjdo! \cdot m$ and for each $y \in R(\mu)$ we have, according to Lemma 9, a vector measure μ^y in $B(\mu, 2m\delta)$ s.t.

$$(1-\delta)\mu(S_i) + \delta \cdot y = \mu^y(S_i), \qquad i = 1, \cdots, k-1. \tag{14}$$

Since $0 < 2m\delta < r$ we have by (13)

$$\|f \circ \mu - f \circ \mu^y\|_{BV} < \varepsilon/2. \tag{15}$$

On the other hand

$$\begin{aligned}
\|f^\delta \circ \mu - f \circ \mu\|_\Omega &= \sum_{i=0}^{k-1} |f^\delta(\mu(S_{i+1})) - f(\mu(s_{i+1})) - f^\delta(\mu(S_i)) + f(\mu(S_i))| \\
&\leqq \frac{1}{a^m} \sum_{i=1}^{k-1} \int_{y \in C} |f((1-\delta)\mu(S_{i+1}) + \delta \cdot y) - f(\mu(S_{i+1})) \\
&\qquad - f((1-\delta)\mu(S_i) + \delta \cdot y) + f(\mu(S_i))|\, d\lambda(y).
\end{aligned}$$

Thus by (14)

$$\|f^\delta \circ \mu - f \circ \mu\|_\Omega$$

$$\leqq \frac{1}{a^m} \sum_{i=0}^{k-1} \int_{y \in C} |f(\mu^y(S_{i+1})) - f(\mu(S_{i+1})) - f(\mu^y(S_i)) + f(\mu(S_i))| \, d\lambda(y)$$

$$+ \frac{1}{a^m} \int_{y \in C} |f((1-\delta)\mu(I) + \delta \cdot y) - f(\mu(I))| \, d\lambda(y)$$

$$+ \frac{1}{a^m} \int_{y \in C} |f(\delta \cdot y)| \, d\lambda(y).$$

The first summand is bounded by $\|f \circ \mu - f \circ \mu^y\|_{BV}$ and according to (15) it is smaller than $\varepsilon/2$. By Lemma 8 there exists δ_1, $0 < \delta_1 < 1$ s.t. for each δ, $0 < \delta < \delta_1$, the last two summands are together smaller than $\varepsilon/2$. Define $\delta_0 = \min(\delta_1, r/2m)$ to complete the proof of Lemma 10.

b. *The condition is necessary*

As in the first part, we first start with the idea of the proof. Assume that $f \circ \mu$ is in pNA; our purpose is to prove that f is continuous at μ. From the definition of pNA there are polynomials $(p_n)_{n=1}^{\infty}$ and vectors of NA^1 measures μ^n s.t. $\|p_n \circ \mu^n - f \circ \mu\|_{BV} \to 0$ as n tends to ∞. Define

$$\hat{R}^n = \{x \in R((\mu, \mu^n)) \mid t_1(x) \in C\}$$

where $t_1(x)$ is the projection of x on $R(\mu)$ and C is defined as in Definition 6. On $\hat{R}^n$ we choose probability measures λ_n and we define for each n and $0 < \delta < 1$, the function f_n^δ on $R(\mu)$ by

$$f_n^\delta(x) = \int_{y \in \hat{R}^n} f((1-\delta)x + \delta \cdot t_1(y)) d\lambda_n(y) - \int_{y \in \hat{R}^n} f(\delta \cdot t_1(y)) d\lambda_n(y).$$

In fact, we will choose the λ_n in such a way that f_n^δ is independent of n and thus we can denote $f^\delta = f_n^\delta$. In order to prove that the above integrals are well defined we prove that f is continuous on $\operatorname{Int} R(\mu)$ (Corollary 15). Then we conclude that f^δ is continuously differentiable on $R(\mu)$. Define for each n and $0 < \delta < 1$ the function p_n^δ on $R(\mu^n)$ by

$$p_n^\delta(x) = \int_{y \in \hat{R}^n} p_n((1-\delta)x + \delta \cdot t_2(y)) d\lambda_n(y) - \int_{y \in \hat{R}^n} p_n(\delta \cdot t_2(y)) d\lambda_n(y),$$

where $t_2(y)$ is the projection of y on $R(\mu^n)$. We prove (Lemma 19) that

$$\|p_n^\delta \circ \mu^n - f^\delta \circ \mu\|_{BV} \xrightarrow[n \to \infty]{} 0 \tag{16}$$

uniformly in $0 < \delta < 1$. In addition we prove (Lemma 20) that for each n

$$\|p_n \circ \mu^n - p_n^\delta \circ \mu^n\|_{BV} \xrightarrow[\delta \to 0]{} 0, \tag{17}$$

and thus (16), (17) and $\|p_n \circ \mu^n - f \circ \mu\|_{BV} \xrightarrow[n\to\infty]{} 0$ imply

$$\|f^\delta \circ \mu - f \circ \mu\|_{BV} \xrightarrow[\delta \to 0]{} 0.$$

We can thus approach $f \circ \mu$ by $f^\delta \circ \mu$ where f^δ is continuously differentiable on $R(\mu)$. Using the fact that the polynomials, with m variables, are dense in† $C^1[R(\mu)]$ we prove (Lemma 22) that one can approach $f \circ \mu$ by $g_n \circ \mu$ where $(g_n)_{n=1}^\infty$ are polynomials on R^m (and so Property I is proved). Now, based on the existence of such $(g_n \circ \mu)_{n=1}^\infty$ we can complete the proof of the second part, as follows.

LEMMA 11. *Any polynomial p on R^m is continuous at μ.*

PROOF. It follows immediately from the fact that BV with the variation norm is a Banach algebra (see proposition 4.5 of [2, p. 29]).

LEMMA 12. *Let $\eta \in (NA^1)^m$. Then, for each $\nu \in B(\eta)$ and for each chain Ω, $\Omega : \emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = I$ of measurable sets there exists a chain Ω^*, $\Omega^* : \emptyset = S_0^* \subseteq S_1^* \subseteq \cdots \subseteq S_k^* = I$ of measurable sets s.t. for each i, $1 \leq i \leq m$,*

$$\nu(S_i) = \eta(S_i^*).$$

(Hence, for each $\nu \in B(\eta)$, $\|f \circ \nu\|_{BV} = \|f \circ \eta\|_{BV}$.)

The proof of this lemma is due to A. Neyman [8].

Now, assume that there are polynomials g_n on R^m with

$$\|g_n \circ \mu - f \circ \mu\|_{BV} \xrightarrow[n\to\infty]{} 0.$$

Then for a given $\varepsilon > 0$ there is an N s.t. $n > N$ implies

$$\|g_n \circ \mu - f \circ \mu\|_{BV} < \varepsilon/3. \tag{18}$$

By (18) and by Lemma 12, for each $\nu \in B(\mu)$ and for each $n > N$

$$\|g_n \circ \nu - f \circ \nu\|_{BV} < \varepsilon/3. \tag{19}$$

Let $n_0 > N$ be fixed. By Lemma 11, there is $\delta > 0$ s.t.

† $C^1[R(\mu)]$ is the set of all continuously differentiable functions f on $R(\mu)$, with the norm

$$\|f\|_1 = \|f\|_0 + \sum_i \|f_i\|_0$$

where f_i is $\partial f/\partial x_i$ in $\operatorname{Int} R(\mu)$, and is the appropriate continuous extension on the boundary, $\|f\|_0 = \max_{x \in R(\mu)} |f(x)|$.

(20) $$\nu \in B(\mu,\delta) \Rightarrow \| g_{n_0}\circ \mu - g_{n_0}\circ \nu \|_{BV} < \varepsilon/3.$$

Therefore by (18), (19) and (20), for any ν in $B(\mu,\delta)$

$$\begin{aligned}\| f\circ \mu - f\circ \nu \|_{BV} &\leqq \| f\circ \mu - g_{n_0}\circ \mu \|_{BV} + \| g_{n_0}\circ \mu - g_{n_0}\circ \nu \|_{BV} + \| g_{n_0}\circ \nu - f\circ \nu \|_{BV} \\ &< \varepsilon,\end{aligned}$$

and the proof of the second part is completed. Notice that Property II follows immediately from (18). It remains now to prove the existence of $(g_n)_{n=1}^{\infty}$ on R^m with $\| g_n\circ \mu - f\circ \mu \|_{BV} \to 0$.

DEFINITION 13. A set function v is said to be absolutely continuous if there is a $\sigma \in NA^1$ s.t. for every $\varepsilon > 0$ there is a $\delta > 0$ obeying for every chain Ω and every subchain Λ of Ω, $\| \sigma \|_{\Lambda} \leqq \delta \Rightarrow \| v \|_{\Lambda} \leqq \varepsilon$ (in this case we write $v \ll \sigma$). The set of all absolutely continuous set functions in BV is denoted by AC.

LEMMA 14. *Let $\xi \in (NA^1)^l$ and let g be a real function on $R(\xi)$. If $g\circ\xi \in AC$ then g is continuous in* $\operatorname{Rel\,Int} R(\xi)$ *(the relative interior of $R(\xi)$).*

PROOF. Let x be in $\operatorname{Rel\,Int} R(\xi)$. $g\circ\xi \in AC$ implies the existence of a measure $\nu \in NA^1$ with $g\circ\xi \ll \nu$. Let $\varepsilon > 0$ be given. There is an $\alpha > 0$ s.t. for each subchain Λ

(21) $$\| \nu \|_{\Lambda} < \alpha \Rightarrow \| g\circ\xi \|_{\Lambda} < \varepsilon/2.$$

Since $x \in \operatorname{Rel\,Int} R(\xi)$ and $R(\xi,\nu)$ is convex, there is $x_1 \in [0,1]$ for which $\bar{x} = (x, x_1)$ is in $\operatorname{Rel\,Int} R(\xi,\nu)$. Applying Lemma 3, we have $\delta > 0$ s.t. for each $\bar{y} \in R(\xi,\nu)$ with $\| \bar{y} - \bar{x} \| < \delta$ there are sets S and T in $\mathscr{C}$, such that

(22) $$\bar{x} = (\xi,\nu)(S), \quad \bar{y} = (\xi,\nu)(T) \quad \text{and} \quad \| (\xi,\nu)(S \triangle T) \| < \alpha.$$

Denote

$$U_{\bar{x}}^{\delta} = \{ \bar{y} \in R(\xi,\nu) \mid \| \bar{y} - \bar{x} \| < \delta \},$$

and let P be the projection of $R(\xi,\nu)$ on $R(\xi)$. $P(U_{\bar{x}}^{\delta})$ contains a neighborhood $U_x^{\delta_1}$ of x (with radius δ_1) in $R(\xi)$. Hence, for each $y \in R(\xi)$ with $\| y - x \| < \delta_1$ there is a $\bar{y}$ in $P^{-1}(y) \cap U_{\bar{x}}^{\delta}$ and there are measurable sets S and T satisfying (22). Therefore

(23) $$x = \xi(S), \quad y = \xi(T) \quad \text{and} \quad \nu(S \triangle T) < \alpha.$$

Define now two subchains Λ_1 and Λ_2 by

$$\Lambda_1 : S \subseteq S \cup T, \qquad \Lambda_2 : T \subseteq S \cup T,$$

and replace Λ in (21) once by Λ_1 and once by Λ_2. Then together with (23) we have

$$|g(\xi(S \cup T)) - g(\xi(S))| < \varepsilon/2,$$

$$|g(\xi(S \cup T)) - g(\xi(T))| < \varepsilon/2.$$

Hence, for each $y \in R(\xi)$ with $\|y - x\| < \delta_1$

$$\begin{aligned} |g(x) - g(y)| &= |g(\xi(S)) - g(\xi(T))| \\ &\leqq |g(\xi(S)) - g(\xi(S \cup T))| + |g(\xi(S \cup T)) - g(\xi(T))| \\ &< \varepsilon. \end{aligned}$$

COROLLARY 15. *The result of Lemma* 14 *is valid if we replace AC by pNA.*

PROOF. This is an immediate consequence of Lemma 14 above and of Corollary 5.3 of [2, p. 36] which asserts that $pNA \subseteq AC$.

DEFINITION 16. For each subset S of any euclidean space we denote by λ_S the Lebesgue measure on the linear manifold spanned by S.

LEMMA 17. *Let A be a compact and convex subset of R^m. Let P be a projection of A on R^l for $l < m$, and assume that $\lambda_{PA}(PA) > 0$. Then, there is a probability measure ν on A s.t. νP^{-1} is the normalized Lebesgue measure on PA.*

PROOF. For every x in PA denote

$$A_x = P^{-1}(\{x\}) \cap A, \qquad \lambda_x = \lambda_{A_x}(A_x).$$

If $\dim PA < \dim A$ then, from the convexity of A, for almost every x in PA (with respect to λ_{PA}) $\lambda_x > 0$. In this case we define a function f on A by:

$$f(a) = \begin{cases} 1/\lambda_x & a \in A_x \text{ and } \lambda_x > 0, \\ 0 & a \in A_x \text{ and } \lambda_x = 0. \end{cases}$$

It is easy to verify that f is λ_A integrable. Define now a measure ν on A as follows: For every measurable subset S of A

$$\nu(S) = \frac{1}{\lambda_{PA}(PA)} \int_{a \in S} f(a) d\lambda_A(a).$$

ν is a normalized measure and νP^{-1} is the normalized Lebesgue measure on PA. In the case where $\dim PA = \dim A$, the measure ν is simply defined by

$$\nu = \frac{1}{\lambda_A(A)} \lambda_A.$$

The proof of the lemma is then complete.

By Lemma 17, for each n there is a normalized measure λ_n defined on $\hat{R}_n$ s.t. $\lambda_n t^{-1}$ is the normalized Lebesgue measure on C. (Recall that $\hat{R}^n = \{x \in R((\mu, \mu^n)) \mid t_1(x) \in C\}$ where $t_1(x)$ is the projection of x on $R(\mu)$, C is a small cube in $\operatorname{Int} R(\mu)$ of the form $[-a/2, a/2]^m + \mu(I)/2$ and

$$\| p_n \circ \mu^n - f \circ \mu \|_{BV} \xrightarrow[n\to\infty]{} 0.)$$

For each n and $0 < \delta < 1$ define on $R(\mu)$ a function f_n^δ by

$$f_n^\delta(x) = \int_{y \in \hat{R}^n} f((1-\delta)x + \delta \cdot t_1(y)) d\lambda_n(y) - \int_{y \in \hat{R}^n} f(\delta \cdot t_1(y)) d\lambda_n(y).$$

By Corollary 15, f_n^δ is well defined. In fact, from the choice of λ_n, f_n^δ is independent of n since

$$f_n^\delta(x) = \int_{y \in t_1(\hat{R}^n)} [f((1-\delta)x + \delta \cdot y) - f(\delta \cdot y)] d\lambda_n t_1^{-1}(y)$$

and thus

$$f_n^\delta(x) = \int_{y \in C} [f((1-\delta)x + \delta \cdot y) - f(\delta \cdot y)] d\lambda(y), \tag{24}$$

where λ is the Lebesgue measure on R^m. We can then write f^δ instead of f_n^δ. By (24) and by Lemma 7 we have now

COROLLARY 18. *For each $0 < \delta < 1$, f^δ is continuously differentiable on $R(\mu)$.*

LEMMA 19. *For every $\varepsilon > 0$ there is an integer N s.t. for each $n > N$ and $0 < \delta < 1$*

$$\| f^\delta \circ \mu - p_n^\delta \circ \mu^n \|_{BV} < \varepsilon.$$

PROOF. Let $0 < \delta < 1$ be given. Let Ω be a chain

$$\Omega : \emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = I, \qquad S_i \in \mathscr{C}.$$

Following the definition of p_n^δ we have

$$\| f^\delta \circ \mu - p_n \circ \mu^n \|_\Omega$$

$$= \sum_{i=0}^{k-1} \Big| \int_{\hat{R}^n} [f((1-\delta)\mu(S_{i+1}) + \delta \cdot t_1(y)) - p_n((1-\delta)\mu^n(S_{i+1}) + \delta \cdot t_2(y)) \tag{25}$$
$$- f((1-\delta)\mu(S_i) + \delta \cdot t_1(y)) + p_n((1-\delta)\mu^n(S_i) + \delta \cdot t_2(y))] d\lambda_n(y) \Big| .$$

For each $y \in \hat{R}^n$ there is a measurable set T_y with $y = (\mu, \mu^n)(T_y)$, and for each i, $0 \leqq i \leqq k-1$,

$$(1-\delta)\chi_{S_{i+1}} + \delta \cdot \chi_{T_y} \geqq (1-\delta)\chi_{S_i} + \delta \cdot \chi_{T_y}.$$

By Lemma 3, there is a subchain Λ

$$\Lambda : T_0^{\delta,y} \subseteq T_1^{\delta,y} \subseteq \cdots \subseteq T_k^{\delta,y}$$

s.t. for each i, $0 \leqq i \leqq k$,

$$(\mu, \mu^n)(T_i^{\delta,y}) = (1-\delta)(\mu, \mu^n)(S_i) + \delta \cdot (t_1(y), t_2(y)).$$

This together with (25) imply

$$\|f^\delta \circ \mu - p_n^\delta \circ \mu^n\|_\Omega$$

$$(26) \quad = \sum_{i=0}^{k-1} \left| \int_{y \in \hat{R}^n} [f(\mu(T_{i+1}^{\delta,y})) - p_n(\mu^n(T_{i+1}^{\delta,y})) - f(\mu(T_i^{\delta,y})) + p_n(\mu^n(T_i^{\delta,y}))] d\lambda_n(y) \right|.$$

The integrand on the right-hand side of (26) is bounded for each $y \in \hat{R}^n$ and $0 < \delta < 1$ by $\|f \circ \mu - p_n \circ \mu^n\|_{BV}$ which tends to 0 as $n \to \infty$. Hence the proof of the lemma is completed.

Lemma 20. *For each n, $\|p_n \circ \mu^n - p_n^\delta \circ \mu^n\|_{BV} \underset{\delta \to \infty}{\longrightarrow} 0$.*

Proof. For each n, $x \in R(\mu^n)$ and $0 < \delta < 1$

$$p_n^\delta(x) = \int_{y \in \hat{R}^n} [p_n((1-\delta)x + \delta \cdot t_2(y)) - p_n(\delta \cdot t_2(y))] d\lambda_n(y).$$

The integrand $p_n((1-\delta)x + \delta \cdot t_2(y)) - p_n(\delta \cdot t_2(y))$ can also be written as $p_n((1-\delta)x) + \delta \cdot Q_n^\delta(t_2(y))$, where $Q_n^\delta(t_2(y))$ is a polynomial in $t_2(y)$ with coefficients which are polynomials in x and δ. Hence

$$p_n^\delta(x) = p_n((1-\delta)x) + \delta \cdot \int_{y \in \hat{R}^n} Q_n^\delta(t_2(y)) d\lambda_n(y).$$

But $q_n^\delta(x) \equiv \int_{y \in \hat{R}^n} Q_n^\delta(t_2(y)) d\lambda_n(y)$ is a polynomial in x and δ, and therefore

$$p_n^\delta(x) = p_n((1-\delta)x) + \delta \cdot q_n^\delta(x).$$

Since $R(\mu^n)$ is a compact set then

$$\|p_n^\delta - p_n\| \underset{\sup \delta \to 0}{\longrightarrow} 0 \qquad \text{on } R(\mu^n)$$

or

$$\|p_n^\delta \circ \mu^n - p_n \circ \mu^n\| \underset{\sup \delta \to 0}{\longrightarrow} 0.$$

Since for a fixed n, each $p_n^\delta(x)$ (as a polynomial in x) has the same degree as $p_n(x)$, and since on any finite-dimensional vector space all the norms are equivalent (here we consider the space of all polynomials q with the same number of variables as in P_n and such that $\deg q \leqq \deg p_n$) then

$$\| p_n^\delta \circ \mu^n - p_n \circ \mu^n \|_{BV} \xrightarrow[\delta \to 0]{} 0$$

and the proof is complete.

LEMMA 21. *For each k there is a continuously differentiable function g_k on $R(\mu)$ s.t.*

$$\| g_k \circ \mu - f \circ \mu \|_{BV} \xrightarrow[k \to \infty]{} 0.$$

PROOF. Since

$$\| p_n \circ \mu^n - f \circ \mu \|_{BV} \xrightarrow[n \to \infty]{} 0, \tag{27}$$

by Lemma 20, for each n

$$\| p_n^\delta \circ \mu^n - p_n \circ \mu^n \|_{BV} \xrightarrow[\delta \to 0]{} 0. \tag{28}$$

By Lemma 19

$$\| f^\delta \circ \mu - p_n^\delta \circ \mu^n \|_{BV} \xrightarrow[n \to \infty]{} 0 \qquad \text{uniformly in } 0 < \delta < 1. \tag{29}$$

By (27), (28) and (29) we get

$$\| f^\delta \circ \mu - f \circ \mu \|_{BV} \xrightarrow[\delta \to 0]{} 0. \tag{30}$$

Since f^δ is continuously differentiable on $R(\mu)$ (Corollary 18), the proof is complete.

LEMMA 22. *For each k there is a polynomial q_k on R^m s.t.*

$$\| q_k \circ \mu - f \circ \mu \|_{BV} \xrightarrow[k \to \infty]{} 0.$$

PROOF. The space $(C^1(R(\mu)), \| \ \|_1)$ is defined in the previous footnote. Lemma 7.4 in [2, p. 42] asserts that the polynomials are dense in $C^1(R(\mu))$. (The essence of the proof is given in Courant and Hilbert [3, p. 68].) Therefore, for each $g \in C^1(R(\mu))$ there is a sequence of polynomials $(q_n)_{n=1}^{\infty}$ on R^m for which

$$\| q_n - g \|_1 \xrightarrow[n\to\infty]{} 0.$$

Inequality (7.5) of [2, p. 43] asserts that

$$\| g \circ \mu \|_{BV} \leqq \| g \|_1 \sum_{i=1}^{m} \mu_i(I) \leqq m \cdot \| g \|_1 .$$

Hence

$$\| q_n \circ \mu - g \circ \mu \|_{BV} \leqq m \cdot \| q_n - g \|_1 \xrightarrow[n\to\infty]{} 0.$$

Since we can choose $(g_k)_{k=1}^{\infty}$ on $R(\mu)$ which satisfy the conditions of Lemma 21, there is for each k a sequence of polynomials q_n^k on R^m for which

$$\| q_n^k \circ \mu - g_k \circ \mu \|_{BV} \xrightarrow[n\to\infty]{} 0. \tag{31}$$

(31) together with Lemma 21 imply Lemma 22 as well as the second part of the main theorem. We thus have proved the main theorem for the case where μ has a full dimension.

In the general case, let us assume that $\dim R(\mu) = l$ and $l < m$. Since $0 \in R(\mu)$, there is a linear mapping ψ from R^m to R^l which is 1-1 on the subspace $M(\mu)$ spanned by $R(\mu)$. Let $\tau = \psi|_{M(\mu)}$. For each $S \in \mathscr{C}$ define a vector $\xi(S)$ in R^l by $\xi(S) = \tau\mu(S)$. Thus a vector ξ of l NA^1 measures is defined and ξ has a full dimension. τ induces a 1-1 mapping $\hat{\tau} : (B(\mu), \| \ \|_m) \to (B(\xi), \| \ \|_l)$ such that for each $\mu' \in B(\mu)$

$$\hat{\tau}(\mu') = \xi' \Leftrightarrow \xi'(S) = \tau(\mu'(S)) \qquad \forall S \in \mathscr{C}.$$

Define on $\tau(R(\mu))$ a function g by

$$g(x) = f(\tau^{-1}x).$$

Then $g \circ \xi = f \circ \mu$ and for each $\mu' \in B(\mu)$

$$g \circ (\hat{\tau}\mu') = f \circ \mu'. \tag{32}$$

Lemma 23. *f is continuous at μ if and only if g is continuous at ξ.*

Proof. Assume that f is continuous at μ. Let $\varepsilon > 0$, and choose a $\delta > 0$ satisfying

$$\mu' \in B(\mu, \delta) \Rightarrow \| f \circ \mu - f \circ \mu' \|_{BV} < \varepsilon. \tag{33}$$

Let $0 < \alpha < \delta / \| \tau^{-1} \|$. For each $\xi' \in B(\xi, \alpha)$

$$\| \hat{\tau}^{-1} \xi - \hat{\tau}^{-1} \xi' \|_m \leqq \| \hat{\tau}^{-1} \| \cdot \| \xi - \xi' \|_l < \delta.$$

Therefore $\hat{\tau}^{-1} \xi' \in B(\mu, \delta)$, and by (32) and (33) we have

$$\| g \circ \xi - g \circ \xi' \|_{BV} < \varepsilon,$$

i.e., g is continuous at ξ. The proof of the other direction is similar.

REMARK. Lemma 22 holds also in the case where μ does not have full dimension. Since if $f \circ \mu \in pNA$ then $g \circ \xi \in pNA$ and for the game $g \circ \xi$ Lemma 22 is valid, thus there are polynomials $(q_n)_{n=1}^{\infty}$ on R^l with

$$\| q_n \circ \xi - g \circ \xi \|_{BV} \xrightarrow[n \to \infty]{} 0. \tag{34}$$

Define for every $x \in R^m$, $p_n(x) = q_n(\psi x)$. Since ψ is linear $q_n(\psi x)$ (and hence $P_n(x)$) is a polynomial in x. On the other hand $p_n \circ \mu = q_n \circ \xi$, thus replacing in (34) $q_n \circ \xi$ by $p_n \circ \mu$ and $g \circ \xi$ by $f \circ \mu$ we get $\| p_n \circ \mu - f \circ \mu \|_{BV} \xrightarrow[n \to \infty]{} 0$.

EXAMPLE 24. We present here an alternative proof for the fact that the game v, defined in example 9.4 of [2, p. 78], is not in pNA. v is defined as follows: Let $I = [0, 2]$ and let $\mathscr{C}$ be the σ-field of Borel subsets of I. λ is the Lebesgue measure on I and the measures λ_1, λ_2 in NA^1 are defined by

$$\lambda_1(S) = \lambda(S \cap [0, 1]), \qquad S \in \mathscr{C},$$

$$\lambda_2(S) = \lambda(S \cap [1, 2]), \qquad S \in \mathscr{C}.$$

Let $\mu = \lambda_2 - \lambda_1$ and $v = |\mu|$.

PROPOSITION 26. *v is not in pNA.*

PROOF. Define $f : R^2 \to R^1$ by $f((x_1, x_2)) = |x_2 - x_1|$, then $v = f \circ (\lambda_1, \lambda_2)$. We will show that f is not continuous at (λ_1, λ_2). For each integer n we define λ_2^n and μ^n by

$$\lambda_2^n(S) = \frac{n}{n-1} \lambda(S \cap [1, 2 - 1/n]), \qquad \mu^n = \lambda_1 - \lambda_2^n.$$

It is clear that $(\lambda_1, \lambda_2^n) \in B((\lambda_1, \lambda_2))$ for each n, and

$$\| (\lambda_1, \lambda_2^n) - (\lambda_1, \lambda_2) \|_2 \xrightarrow[n \to \infty]{} 0.$$

But, as will be shown, $\| |\mu| - |\mu^n| \|_{BV} > \frac{1}{2}$, for $n \geqq 3$. For that purpose we define for every $n \geqq 3$

$$A_k^n = \left[1 - \frac{k}{n}, 1 - \frac{k-1}{n}\right], \qquad k = 1, \cdots, n,$$

$$B_k^n = \left[1 + \frac{k-1}{n}, 1 + \frac{k}{n}\right], \qquad k = 1, \cdots, n,$$

$$C^n = [2 - 1/n, 2].$$

Let Ω^n be the chain $\Omega^n : \varnothing = S_0^n \subseteq S_1^n \subseteq \cdots \subseteq S_{2n}^n = I$, where

$$S_1^n = A_1^n \cup C^n, \qquad S_2^n = A_1^n \cup B_1^n \cup C^n,$$

$$S_{2i-1}^n = \left(\bigcup_{t=1}^{i} A_t^n\right) \cup \left(\bigcup_{t=1}^{i-1} B_t^n\right) \cup C^n \quad \text{and} \quad S_{2i}^n = \left(\bigcup_{t=1}^{i} A_t^n\right) \cup \left(\bigcup_{t=1}^{i} B_t^n\right) \cup C^n;$$

$$\| |\mu| - |\mu^n| \|_{\Omega^n} \geqq \sum_{i=1}^{n} \left| |\mu|(S_{2i}^n) - |\mu^n|(S_{2i}^n) - |\mu|(S_{2i-1}^n) + |\mu^n|(S_{2i-1}^n) \right|$$

$$= \sum_{i=1}^{n} \left| \frac{1}{n} - \frac{i}{n(n-1)} - 0 + \frac{n-i}{n(n-1)} \right|$$

$$= \frac{n-2}{n-1} \geqq \frac{1}{2} \qquad (n \geqq 3).$$

Let us mention that for pNA' which is defined in the same way as pNA but with the sup norm (instead of the variation norm used for pNA), we can characterize set functions of the form $f \circ \mu$ in pNA' similarly to the one for pNA. First we define the continuity of f at μ, with respect to the sup norm as follows: For each $\varepsilon > 0$ there is a $\delta > 0$ s.t. for each $\nu \in B(\mu, \delta)$, $\|f \circ \mu - f \circ \nu\|_{\sup} < \varepsilon$. Then, we can prove the following:

PROPOSITION 27. *$f \circ \mu \in pNA'$ if and only if f is continuous at μ (in the* sup *norm).*

The argumentation of the proof is the same as that of the proof of the main theorem, and in fact is much easier. In this way, we also get that for $f \circ \mu \in pNA'$ there exist polynomials p_n on R^m, for which

$$\|p_n \circ \mu - f \circ \mu\| \xrightarrow[\sup]{} 0 \qquad \text{as } n \to \infty.$$

Thus, together with the Stone–Weierstrass theorem we get

PROPOSITION 28. *$f \circ \mu \in pNA'$ iff f is a continuous function on $R(\mu)$.*

Finally a similar characterization for games $f \circ \mu$ in $pNAD$ can be stated (for the definition of the space $pNAD$ see [2, p. 253]). For the space $pNAD$ we use the diagonal variation norm $\|\cdot\|_D$ which is defined to be the limit of $\|\cdot\|_\delta$ as δ tends to zero (for the definition of $\|\cdot\|_\delta$ see [2, p. 262]).

PROPOSITION 29. *Let $f \circ \mu$ be in BV. Then $f \circ \mu$ is in pNAD iff f is continuous at μ (in the diagonal variation norm).*

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